J. DIFFERENTIAL GEOMETRY
75 (2007) 359-386

3-MANIFOLDS WITH YAMABE INVARIANT GREATER THAN THAT OF $\mathbb{RP}^3$

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Abstract

We show that, for all nonnegative integers k, ℓ, m and n, the *Yamabe invariant* of

$$\#k(\mathbb{RP}^3)\#\ell(\mathbb{RP}^2 \times S^1)\#m(S^2 \times S^1)\#n(S^2\tilde{\times}S^1)$$

is equal to the *Yamabe invariant* of $\mathbb{RP}^3$, provided $k + \ell \geq 1$. We then complete the classification (started by Bray and the second author) of all closed 3-manifolds with Yamabe invariant greater than that of $\mathbb{RP}^3$. More precisely, we show that such manifolds are either S^3 or finite connected sums $\#m(S^2 \times S^1)\#n(S^2\tilde{\times}S^1)$, where $S^2\tilde{\times}S^1$ is the nonorientable S^2-bundle over S^1.

A key ingredient is Aubin's Lemma [**3**], which says that if the *Yamabe constant* is positive, then it is strictly less than the *Yamabe constant* of any of its non-trivial finite conformal coverings. This lemma, combined with inverse mean curvature flow and with analysis of the Green's function for the conformal Laplacians on specific finite and normal infinite Riemannian coverings, will allow us to construct a family of nice test functions on the finite coverings and thus prove the desired result.

1. Introduction

It is well known that, for a closed smooth n-manifold M with $n \geq 3$, a Riemannian metric is Einstein if and only if it is a critical point of the normalized Einstein-Hilbert functional E on the space $\mathcal{M}(M)$ of all Riemannian metrics on M

$$E : \mathcal{M}(M) \to \mathbb{R}, \quad g \mapsto E(g) = \frac{\int_M R_g \, dV_g}{\mathrm{Vol}_g(M)^{(n-2)/n}}.$$

Here, R_g, dV_g and $\mathrm{Vol}_g(M)$ denote respectively the scalar curvature, the volume form of the metric g, and the volume of (M, g). Next, we give the definition of a natural differential-topological invariant $Y(M)$ of M,

The first author was supported in part by the Grants-in-Aid for Scientific Research (C), Japan Society for the Promotion of Science, No. 16540059. The second author was supported in part by FCT-Portugal, Grant BD/893/2000.
Received 11/16/2004.

called the *Yamabe invariant* or *σ-invariant*. Denote the conformal class of a Riemannian metric g by

$$C = [g] := \{\mathrm{e}^{2f} \cdot g \mid f \in C^\infty(M)\}$$

and the space of all conformal structures by $\mathcal{C}(M)$. The Yamabe invariant $Y(M)$ is defined by

$$Y(M) := \sup_{C \in \mathcal{C}(M)} \inf_{g \in C} E(g).$$

Contrarily to what was stated in [**7**], this invariant was introduced independently by O. Kobayashi [**15**] and Schoen [**30**].

To put this definition in context, first recall that E is neither bounded from above nor from below for any M, so a minimization method cannot be applied to E. Nevertheless, the study of the second variation of E done in [**16**, **31**] (cf. [**6**]) leads naturally, as we will argue next, to the mini-max definition of the Yamabe invariant.

Let $\mathcal{M}_1(M)$ denote the unit volume metrics of M. Due to results of Obata [**24**] and Aubin [**4**], an Einstein metric h in $\mathcal{M}_1(M)$ globally minimizes E in $\mathcal{M}_1(M) \cap [h]$. Moreover, if $(M, [h])$ is not conformally equivalent to $(S^n, [g_0])$, E is strictly globally minimized among variations in $T_h(\mathcal{M}_1(M) \cap [h])$. Here, g_0 is the canonical metric of constant curvature one on the standard n-sphere S^n. On the other hand, if we consider the infinite dimensional subspace V consisting of infinitesimal deformations orthogonal to both $T_h[h]$ and the subspace of infinitesimal deformations arising from the diffeomorphism group $\mathrm{Diff}(M)$ of M, E is locally maximized among variations in V modulo a finite dimensional subspace (possibly the zero subspace). These observations roughly imply that critical points are always saddle points over the moduli space $\mathcal{M}_1(M)/\mathrm{Diff}(M)$.

The restriction of E to any given conformal class $[g]$ is always bounded from below. Hence, we can consider the following conformal invariant

$$\begin{aligned} &Y(M, [g]) \\ &:= \inf\{E(\tilde{g}) \mid \tilde{g} = u^{4/(n-2)} \cdot g \in [g],\ u > 0 \text{ in } C^\infty(M)\} \\ &= \inf\left\{ Q_g(u) := \frac{\int_M (\frac{4(n-1)}{n-2}|\nabla u|^2 + R_g u^2) dV_g}{\left(\int_M u^{2n/(n-2)}\ dV_g\right)^{(n-2)/n}} \;\middle|\; u > 0 \text{ in } C^\infty(M) \right\}, \end{aligned}$$

which is called the *Yamabe constant* of $[g]$. A remarkable theorem [**37**, **4**, **5**, **29**] (cf. [**22**, **35**]) of Yamabe, Trudinger, Aubin, and Schoen asserts that each conformal class $[g]$ contains metrics $\check{g}$, called *Yamabe metrics*, which realize the minimum

$$Y(M, [g]) = E(\check{g}).$$

These metrics must have constant scalar curvature

$$R_{\check{g}} = Y(M, [g]) \cdot \mathrm{Vol}_{\check{g}}(M)^{-2/n}.$$

Aubin [**4**] (cf. [**5**]) also proved that

$$Y(M,C) \leq Y(S^n,[g_0]) = n(n-1)\mathrm{Vol}_{g_0}(S^n)^{2/n}$$

for any $C \in \mathcal{C}(M)$. This implies that

$$Y(M) = \sup_{C\in\mathcal{C}(M)} Y(M,C) \leq Y(S^n) = n(n-1)\mathrm{Vol}_{g_0}(S^n)^{2/n}.$$

There is a substantial body of work on Yamabe invariants and we will briefly review some significant results (see the surveys [**1**, **17**, **21**, **36**] for example). The Yamabe invariant $Y(M)$ is positive if and only if M carries a metric of positive scalar curvature. Some classification theorems for manifolds with positive Yamabe invariant have been obtained in [**8**, **9**, **32**, **33**, **36**] which show, for instance, that the n-torus T^n has $Y(T^n) = 0$. There are also some useful surgery theorems [**15**, **25**, **26**] for Yamabe invariants. LeBrun [**20**] proved that, for compact complex surfaces, the Yamabe invariant is positive, zero, or negative, if and only if the Kodaira dimension is $-\infty, 0$ or 1, or 2, respectively. LeBrun and collaborators [**10**, **14**, **18**, **19**] also computed the Yamabe invariants for a large class of 4-manifolds, including Kähler surfaces X with $Y(X) < 0$ and $0 < Y(X) < Y(S^4)$. In [**21**], it is constructed a Kähler surface which admits Einstein metrics but the Yamabe invariant is not achieved by any Yamabe metric.

Regarding 3-manifolds, the following conjecture by Schoen [**31**] is one of the most fundamental unsolved problems related to Yamabe invariants.

Conjecture. If M is a closed 3-manifold which admits a constant curvature metric h, then

$$Y(M) = E(h).$$

Note that, for an Einstein metric h on an n-manifold N ($n \geq 3$), we have that $E(h) = Y(N,[h])$. If $E(h) > 0$, this comes from Obata's Theorem [**24**], while the nonpositive case comes from the uniqueness (up to a normalizing constant) of Yamabe metrics. Setting

$$Y_1 := Y(S^3) = 6 \cdot \mathrm{Vol}_{g_0}(S^3)^{2/3}, \quad Y_k := \frac{Y_1}{k^{2/3}} \quad \text{for} \quad k \in \mathbb{N},$$

this conjecture implies, in particular, that if S^3/Γ is a smooth quotient by a finite group Γ of order k, then $Y(S^3/\Gamma) = Y_k$. This conjecture is known to be true for any closed manifold uniformized by T^3 and for S^3. In [**7**], Bray and the second author showed that it is also true for $S^3/\mathbb{Z}_2 = \mathbb{RP}^3$.

Theorem 1.1 (Bray–Neves)**.** *Let M be a closed 3-manifold satisfying the following conditions:*

(i) *M is not diffeomorphic to S^3;*
(ii) *M does not contain any non-separating embedded 2-sphere;*

(iii) *M does not contain any two-sided real projective plane $\mathbb{RP}^2$.*

Then,

$$Y(M,[g]) \leq Y_2$$

for any conformal class $[g]$. Moreover, the equality holds if and only if (M,g) is conformally equivalent to $\mathbb{RP}^3$ with the metric h_0 of constant curvature one. In particular, $Y(M) \leq Y_2$.

Remark 1.2.

(a) The result concerning the conformal equivalence to $(\mathbb{RP}^3,[h_0])$ was not investigated in [**7**]. However, it follows easily from the characterization of the equality condition in the *Riemannian Penrose Inequality* [**13**, Section 8].

(b) Condition (iii) in Theorem 1.1 is vacuous when M is orientable. A more general condition was stated in [**7**] and it includes the case where M is $\mathbb{RP}^2 \times S^1$. The necessity of such condition is addressed later.

As a consequence, they obtained the following corollaries.

Corollary 1.3.

$$Y(\mathbb{RP}^3) = Y_2,\ Y(\mathbb{RP}^2 \times S^1) = Y_2.$$

Corollary 1.4. *The only prime 3-manifolds with Yamabe invariant greater than Y_2 are S^3, $S^2 \times S^1$ and $S^2 \tilde{\times} S^1$ (the nonorientable S^2 bundle over S^1).*

All these manifolds have Yamabe invariant equal to Y_1. The fact that $Y(S^2 \times S^1) = Y_1$ was proven independently by Kobayashi [**15**] and Schoen [**31**].

We now describe, in an informal way, the idea behind the proof of Theorem 1.1. Most of the definitions can be found in Section 3 but, for a more elaborated exposition, the reader is encouraged to see [**7**, Sections 3 and 4]. All the statements regarding the inverse mean curvature flow are proven in [**13**].

The general approach was to construct a nice test function, not on (M,g), but instead on a conformally equivalent scalar-flat, asymptotically flat manifold $(M - \{p\}, g_{AF})$, where p is a point in M. Bray and Neves observed that if this procedure is done for $(\mathbb{RP}^3, h_0)$, then the optimal test function u_0 on the corresponding scalar-flat, asymptotically flat manifold (which is the spacial Schwarzschild manifold), has the same level sets as the inverse mean curvature flow $\{\widehat{\Sigma}_t\}_{t\geq 0}$ starting on the unique outermost minimal surface.

The main idea was to construct a test function u on the asymptotically flat manifold $M - \{p\}$ having the same level sets as the inverse mean curvature flow $\{\Sigma_t\}_{t\geq 0}$ starting on a connected component of the

outermost minimal surface. For each time t, u assumes on Σ_t the value that u_0 assumes on $\widehat{\Sigma}_t$.

They showed next that if the Hawking quasi-local mass is nondecreasing, which happens precisely if $\chi(\Sigma_t) \leq 2$ for all t, then $E(u^4 g_{AF}) \leq Y_2$. The topological assumptions on Theorem 1.1 assure that, if the initial condition has Euler characteristic less or than or equal to two, then this condition is preserved by the inverse mean curvature (see Theorem 3.4 and Lemma 3.5) and so the result follows.

A question left open in [**7**] was how to compute $Y(\mathbb{RP}^3\#(S^2 \times S^1))$ or $Y((\mathbb{RP}^2 \times S^1)\#(\mathbb{RP}^2 \times S^1))$. The problem with the above idea is that, in these cases, it is impossible to guarantee the monotonicity of the Hawking quasi-local mass. The lack of monotonicity is a serious issue. As a matter of fact, we point out that the reason why this idea does not work for $S^2 \times S^1$ (which has the Yamabe invariant Y_1), is precisely because the Hawking quasi-local mass does not remain monotone (for some interval of time, the flow might become a disconnected union of 2-spheres). In this paper, we find a way to overcome this difficulty.

Acknowledgements. The first author would like to thank Richard Schoen and Rafe Mazzeo for helpful discussions during his short visit to Stanford University in March 2003. The second author is thankful to Steve Kerkchoff for many helpful discussions. He would also like to express his gratitude to his advisor, Richard Schoen, for all of his invaluable guidance.

2. Main results

We now describe the main results of this paper. We denote the connected sum of k copies of a closed manifold M by $\#kM$ and the nonorientable S^2-bundle over S^1 by $S^2 \tilde{\times} S^1$.

The first result is a generalization of Theorem 1.1.

Theorem 2.1.

(1) *Let M be a closed 3-manifold satisfying the following conditions:*
   (i) *M is not diffeomorphic to S^3;*
   (ii) *M does not contain any non-separating embedded 2-sphere;*
   (iii) *M does not contain any two-sided real projective plane $\mathbb{RP}^2$.*
   If L is a closed 3-manifold which has a non-separating 2-sphere, then
$$Y(M\#L, [g]) < Y_2$$
   for any conformal class $[g]$. In particular, $Y(M\#L) \leq Y_2$.

(2) *Let M be a closed 3-manifolds with a two-sided $\mathbb{RP}^2$, then*
$$Y(M, [g]) < Y_2$$
   for any conformal class $[g]$. In particular, $Y(M) \leq Y_2$.

The next corollary follows easily from Theorem 1.1 and Theorem 2.1.

Corollary 2.2. *For any closed* 3*-manifold* L *which is not diffeomorphic to* S^3*,*

$$Y(\mathbb{RP}^3\#L,[g])<Y_2 \quad \textit{and} \quad Y((\mathbb{RP}^2\times S^1)\#L,[\tilde{g}])<Y_2$$

for any conformal classes $[g]$ *and* $[\tilde{g}]$*, respectively. In particular,*

$$Y(\mathbb{RP}^3\#L)\le Y_2 \quad \textit{and} \quad Y((\mathbb{RP}^2\times S^1)\#L)\le Y_2.$$

Given any two manifolds M_1 and M_2 with nonnegative Yamabe invariants, Kobayashi [**15**] proved the following fundamental inequality for the connected sum $M_1\#M_2$:

$$Y(M_1\#M_2)\ge \min\{Y(M_1),\, Y(M_2)\}.$$

This inequality, Theorem 2.1, and the fact that

$$Y(S^2\times S^1)=Y_1=Y(S^2\tilde{\times}S^1) \tag{1}$$

imply the next corollary.

Corollary 2.3. *For any nonnegative integers* k,ℓ,m,n *with* $k+\ell\ge 1$*, we have*

$$Y(\#k\mathbb{RP}^3\#\ell(\mathbb{RP}^2\times S^1)\#m(S^2\times S^1)\#n(S^2\tilde{\times}S^1))=Y_2.$$

Finally, we obtain the following classification.

Corollary 2.4. *The only closed* 3*-manifolds with Yamabe invariant greater than* Y_2 *are*

$$S^3 \quad \textit{and} \quad \#k(S^2\times S^1)\#\ell(S^2\tilde{\times}S^1),$$

where k *and* ℓ *are any nonnegative integers with* $k+\ell\ge 1$*.*

Proof. This follows from the Prime Decomposition Theorem, which says that any closed 3-manifold, not diffeomorphic to S^3, can be decomposed as

$$M\#k(S^2\times S^1)\#\ell(S^2\tilde{\times}S^1)\#mQ$$

for some nonnegative integers k,ℓ, and m which is either 0 or 1. Here, M satisfies the conditions (ii) and (iii) of Theorem 2.1, and Q is a connected sum of prime 3-manifolds, each of which has a two-sided $\mathbb{RP}^2$ (see [**12**]). Theorem 2.1 implies that M is S^3 and that $m=0$. The statement follows from Kobayashi's inequality combined with (1). q.e.d.

3. Tool box review

The rest of the paper is devoted to the proof of Theorem 2.1, and so we start by briefly reviewing the indispensable key results. The main references are [**7**], [**13**], and [**22**] for the first two subsections and [**3**] for the last subsection.

3.1. Yamabe constants of asymptotically flat 3-manifolds. For any closed conformal 3-manifold (M, C) with positive Yamabe constant, there exists a scalar-flat, asymptotically flat metric $g_{AF} \in C$ on $M - \{\text{point}\}$. It turns out that this asymptotically flat manifold is more convenient for studying Yamabe constants. This has played a central role in [**29**], [**7**], and it will also be important in our approach.

Definition 3.1. A Riemannian 3-manifold (N, h) is said to be *asymptotically flat* if there exists a compact set $K \subset N$ such that $N - K$ is diffeomorphic to $\mathbb{R}^3 - \{|x| \leq 1\}$ and that, in the coordinate chart defined by this diffeomorphism, the metric $h = \sum_{i,j} h_{ij}(x) dx^i dx^j$ satisfies the following:

$$h_{ij} = \delta_{ij} + O(|x|^{-1}), \quad h_{ij,k} = O(|x|^{-2}), \quad h_{ij,kl} = O(|x|^{-3}).$$

Let (M, g) be a closed 3-manifold with positive scalar curvature. Define the *conformal Laplacian* to be

$$L_g := -8\Delta_g + R_g,$$

and consider the normalized Green's function G_p for L_g with a pole at $p \in M$, that is,

$$L_g G_p = c_0 \cdot \delta_p \quad \text{on} \quad M \quad \text{and} \quad \lim_{q \to p} \operatorname{dist}(p, q) G_p(q) = 1.$$

Here, $c_0 > 0$ and δ_p stand respectively for a specific universal positive constant and the Dirac δ-function at p. The maximum principle implies that G_p is positive on $M^* := M - \{p\}$, and hence $g_{AF} := G_p^4 g$ defines a scalar-flat, asymptotically flat metric on M^* (see [**22**, Section 6]).

For the noncompact manifold (M^*, g_{AF}), the Yamabe constant is defined by

$$\begin{aligned} Y(M^*, [g_{AF}]) &:= \inf \left\{ \frac{\int_{M^*} 8|\nabla u|^2 dV_{g_{AF}}}{\left(\int_{M^*} u^6 dV_{g_{AF}}\right)^{1/3}} \,\middle|\, u \not\equiv 0 \text{ in } C_c^\infty(M^*) \right\} \\ &= \inf \left\{ \frac{\int_{M^*} 8|\nabla u|^2 dV_{g_{AF}}}{\left(\int_{M^*} u^6 dV_{g_{AF}}\right)^{1/3}} \,\middle|\, u \not\equiv 0 \text{ in } W^{1,2}(M^*, g_{AF}) \right\}. \end{aligned}$$

From the Sobolev embedding theorem for asymptotically flat manifolds, we note that

$$W^{1,2}(M^*, g_{AF}) \hookrightarrow L^6(M^*, g_{AF}).$$

It is readily seen that $Y(M^*, [g_{AF}]) = Y(M, [g])$.

3.2. Inverse mean curvature flow. Like in [**7**], the inverse mean curvature flow developed by Huisken and Ilmanen in [**13**] will be a crucial tool to prove Theorem 2.1. The main difference this time is its application to a *family* of asymptotically flat manifolds *with boundary*. We now introduce some of their terminology and present the main results of [**13**] that will be necessary for our purposes.

In this subsection, (N, h) will always denote an asymptotically flat 3-manifold with boundary ∂N consisting of a union of minimal surfaces. For a C^1 surface Σ contained in N, $H \in L^1_{\mathrm{loc}}(\Sigma)$ is called the *weak mean curvature* of Σ provided

$$\int_\Sigma \mathrm{div}_N(X)\, dA_h = \int_\Sigma H\langle X, \nu\rangle\, dA_h$$

for all vector fields X with compact support on N, where ν is the exterior unit normal vector. If Σ is smooth, $-H\nu$ coincides with the usual mean curvature of Σ.

For any surface Σ of (N, h), $\left|\Sigma\right| = \left|\Sigma\right|_h$ stands for the 2-dimensional Hausdorff measure of Σ. A precompact set E of N is called a *minimizing hull in N* if it minimizes area on the outside, that is,

$$|\partial^* E| \leq |\partial^* F|$$

for any F containing E such that $F - E \subset\subset N$, where $\partial^* F$ is the reduced boundary of F.

Huisken and Ilmanen [**13**, Sections 3 and 6] proved the following existence result for inverse mean curvature flow.

Theorem 3.2 (Existence)**.** *Let E_0 be an open precompact minimizing hull in N with smooth boundary ∂E_0. Then, there exists a precompact locally Lipschitz ϕ on N satisfying*:

(i) *For $t \geq 0$, $\Sigma_t := \partial\{\phi < t\}$ defines an increasing family of $C^{1,\alpha}$ surfaces $(0 < \alpha < 1)$ such that $\Sigma_0 = \partial E_0$;*
(ii) *For almost all $t \geq 0$, the weak mean curvature of Σ_t is $|\nabla\phi|_{\Sigma_t}$;*
(iii) *For almost all $t \geq 0$,*

$$|\nabla\phi|_{\Sigma_t} \neq 0 \quad \textit{for almost all} \;\; x \in \Sigma_t$$

(*with respect to the surface measure*) *and*

$$\left|\Sigma_0\right| e^t \leq \left|\Sigma_t\right| \leq \left(\left|\Sigma_0\right| + \left|\partial N\right|\right) e^t \quad \textit{for all} \;\; t \geq 0.$$

When ϕ is smooth with nonvanishing gradient, (ii) implies that the surfaces $\{\phi = t\}_{t\geq 0}$ are a solution to inverse mean curvature flow with initial condition ∂E_0 because the velocity vector for such deformation is given by

$$|\nabla\phi|^{-1}\nu,$$

where ν is the exterior unit normal to $\{\phi = t\}$.

Definition 3.3. For a compact C^1 surface Σ with weak mean curvature H in $L^2(\Sigma)$, the *Hawking quasi-local mass* is defined to be

$$m_H(\Sigma) := \sqrt{\frac{\left|\Sigma\right|}{(16\pi)^3}}\left(16\pi - \int_\Sigma H^2 dA_h\right).$$

For the rest of this subsection, assume that E_0 is an open connected precompact minimizing hull in N with smooth boundary ∂E_0, and that $\{\Sigma_t\}_{t\geq 0}$ is a weak solution for the inverse mean curvature flow with initial condition $\Sigma_0 = \partial E_0$.

Under the condition that (N, h) has nonnegative scalar curvature, we have the following remarkable property for the weak inverse mean curvature flow [**13**].

Theorem 3.4 (Geroch Monotonicity)**.** *Let (N, h) be an asymptotically flat 3-manifold with nonnegative scalar curvature. The Hawking quasi-local mass $m_H(\Sigma_t)$ is a nondecreasing function of t provided $\chi(\Sigma_t) \leq 2$ for all t.*

The monotonicity of the Hawking quasi-local mass will be of major importance in proving Theorem 2.1. The next Lemma gives topological conditions on N that guarantee this monotonicity.

Lemma 3.5. *Assume that besides the assumption in Theorem 3.4, N does not contain any non-separating embedded 2-sphere and any two-sided real projective plane. Then,*

$$\chi(\Sigma_t) \leq 2$$

for all $t \geq 0$, provided that ∂E_0 is connected.

Proof. Let ϕ be the function defining the weak solution $\{\Sigma_t\}_{t\geq 0}$. According to [**13**, Sections 4 and 6], the connectedness of E_0 implies that both $\{\phi < t\}$ and $\{\phi > t\}$ remain connected for all t.

It is also proven in [**13**] that, for each $t > 0$, Σ_t can be approximated by earlier surfaces Σ_s for which $\nabla\phi$ never vanishes. Hence, without loss of generality, we can assume that

$$\Sigma_t = \{\phi = t\} = \partial\{\phi > t\} = \partial\{\phi < t\}.$$

The connectedness of the two sets on the right of the above expression implies that Σ_t is two-sided. We now show that $\chi(\Sigma_t) > 2$ is impossible.

If Σ_t contains a 2-sphere, then this 2-sphere has to bound a connected region, and hence $\chi(\Sigma_t) = 2$ because $\{\phi < t\}$ is connected. If Σ_t contains a real projective plane, then it has to be one-sided. This contradicts that Σ_t is two-sided. q.e.d.

3.3. Aubin's lemma. The next Lemma, due to Aubin [**3**], is a key tool to prove Theorem 2.1.

Lemma 3.6 (Aubin)**.** *Let (M, g) be a closed n-manifold with positive Yamabe constant and (M_k, g_k) a non-trivial finite k-fold Riemannian covering (i.e., $k \geq 2$). Then,*

$$Y(M, [g]) < Y(M_k, [g_k]).$$

Remark 3.7. Aubin's proof assumes that M_k is a normal covering of M (cf. [**3**, Theorem 6]). We have to adapt its proof in order to remove this condition.

Proof. Let u be a normalized positive function that minimizes the Yamabe quotient Q_{g_k}, that is,

$$Q_{g_k}(u) = E(u^{4/(n-2)}g_k) = Y(M_k, [g_k]) \quad \text{and} \quad \int_{M_k} u^{2n/(n-2)}\, dV_{g_k} = 1.$$

For any $x \in M_k$, set

$$\{x_1, \cdots, x_k\} := \mathcal{P}^{-1}\big(\mathcal{P}(x)\big) \quad \text{with} \quad x_1 = x,$$

where $\mathcal{P}$ denotes the k-fold covering map. Define positive functions v and $v^{\langle p\rangle}$ (for each $p > 0$) on M_k by

$$v(x) := \sum_{i=1}^{k} u(x_i) \quad \text{and} \quad v^{\langle p\rangle}(x) := \sum_{i=1}^{k} u(x_i)^p.$$

For any *evenly covered* open set U $(\subset M)$ for $\mathcal{P}$, set

$$\{U_1, \cdots, U_k\} := \mathcal{P}^{-1}(U) \ (\subset M_k).$$

Moreover, all the sets are isometric and so we can find, for instance, k isometries

$$\gamma_i : U_1 \longrightarrow U_i \quad i = 1, \cdots, k$$

satisfying $\mathcal{P} \circ \gamma_i = \mathcal{P}$. As a result, it is possible to express v and $v^{\langle p\rangle}$ on U_1 as

$$v = \sum_{i=1}^{k} u \circ \gamma_i, \quad v^{\langle p\rangle} := \sum_{i=1}^{k} (u \circ \gamma_i)^p,$$

and hence $v, v^{\langle p\rangle} \in C^\infty(M_k)$.

We can use the local expression described above in order to find k distinct isometries

$$\gamma_i : M_k - \mathcal{S} \longrightarrow M_k - \mathcal{S}, \quad i = 1, \cdots, k,$$

where $\mathcal{S}$ is a piecewise smooth compact $(n-1)$-submanifold such that $\mathcal{P}^{-1}(\mathcal{P}(\mathcal{S})) = \mathcal{S}$. Hence, if we set $u_i := u \circ \gamma_i$, we have that on $M_k - \mathcal{S}$

$$v = \sum_{i=1}^{k} u_i \quad \text{and} \quad v^{\langle p\rangle} = \sum_{i=1}^{k} u_i^p,$$

from which it follows that

$$\int_{M_k} v^{\langle p\rangle}\, dV_{g_k} = k \int_{M_k} u^p\, dV_{g_k}$$

and

$$L_k v = Y(M_k, [g_k]) v^{\langle (n+2)/(n-2)\rangle} = Y(M_k, [g_k]) \sum_{i=1}^{k} u_i^{(n+2)/(n-2)},$$

where $L_k := -\frac{4(n-1)}{n-2}\Delta_{g_k} + R_{g_k}$ is the conformal Laplacian with respect to g_k.

Define v_0 to be the function on M whose lift to M_k is v. Then,

$$
\begin{aligned}
(2) \qquad Q_g(v_0) &= k^{-2/n} Q_{g_k}(v) \\
&= k^{-2/n} \frac{\displaystyle\int_{M_k} v(L_k v)\, dV_{g_k}}{\left(\displaystyle\int_{M_k} v^{2n/(n-2)}\, dV_{g_k}\right)^{(n-2)/n}} \\
&= k^{-2/n} Y(M_k, [g_k]) \frac{\displaystyle\int_{M_k} v \left(\sum_{i=1}^{k} u_i^{(n+2)/(n-2)}\right) dV_{g_k}}{\left(\displaystyle\int_{M_k} v^{2n/(n-2)}\, dV_{g_k}\right)^{(n-2)/n}}.
\end{aligned}
$$

Using Hölder's inequality twice and the following strict inequality

$$
\left(\sum_{i=1}^{k} a_i^p\right)^{1/p} < \sum_{i=1}^{k} a_i \qquad \text{for} \quad k \geq 2,\ p > 1,\ a_i > 0\ (i = 1, \ldots, k),
$$

we obtain that

$$
\begin{aligned}
(3) \qquad & \int_{M_k} v \left(\sum_{i=1}^{k} u_i^{(n+2)/(n-2)}\right) dV_{g_k} \\
&\leq \int_{M_k} v \left(\sum_{i=1}^{k} u_i^{2n/(n-2)}\right)^{2/n} \left(\sum_{i=1}^{k} u_i^{n/(n-2)}\right)^{(n-2)/n} dV_{g_k} \\
&< \int_{M_k} v^2 \left(\sum_{i=1}^{k} u_i^{2n/(n-2)}\right)^{2/n} dV_{g_k} \\
&\leq \left(\int_{M_k} v^{2n/(n-2)}\, dV_{g_k}\right)^{(n-2)/n} \left(\int_{M_k} \sum_{i=1}^{k} u_i^{2n/(n-2)} dV_{g_k}\right)^{2/n} \\
&= \left(\int_{M_k} v^{2n/(n-2)}\, dV_{g_k}\right)^{(n-2)/n} \left(\int_{M_k} v^{\langle 2n/(n-2)\rangle} dV_{g_k}\right)^{2/n} \\
&= k^{2/n} \left(\int_{M_k} v^{2n/(n-2)}\, dV_{g_k}\right)^{(n-2)/n}.
\end{aligned}
$$

It follows from (2) and (3) that

$$
Q_g(v_0) < Y(M_k, [g_k])
$$

and hence

$$
Y(M, [g]) < Y(M_k, [g_k]).
$$

q.e.d.

4. Overall strategy

In this section we describe our approach to Theorem 2.1-(1) in the simple case that $L = S^2 \times S^1$. Let

$$N := M\#(S^2 \times S^1),$$

where M is a closed 3-manifold satisfying conditions (i), (ii), and (iii) in Theorem 2.1. Without loss of generality, we may assume that $Y(N, [g]) > 0$ and that g is a Yamabe metric with $\mathrm{Vol}_g(N) = 1$. Note that $R_g \equiv Y(N, [g]) > 0$.

Consider Riemannian coverings of (N, g) defined in the following way. Take a smooth loop c in $S^2 \times S^1$ whose homotopy class $\langle c \rangle$ generates $\pi_1(S^2 \times S^1)$. We regard the loop c as a loop in N. Let (N_k, g_k) and (N_∞, g_∞) denote, respectively, the k-fold Riemannian covering and the normal infinite Riemannian covering associated to $\langle c \rangle$. Note that, topologically,

$$N_k = \#kM\#(S^2 \times S^1), \qquad N_\infty = \#_1^\infty M\#(S^2 \times \mathbb{R}),$$

and that the covering $N_\infty \to N_k$ is also normal. From Aubin's Lemma 3.6 we have that, for all $k \geq 2$,

$$Y(N, [g]) < Y(N_k, [g_k]),$$

and this implies that

$$Y(N, [g]) < \limsup_{k\to\infty} Y(N_k, [g_k]).$$

The strict inequality comes from the fact that we can choose an increasing subsequence of $\{Y(N_k, [g_k])\}_{k\geq 1}$, for instance, $\{Y(N_{2^m}, [g_{2^m}])\}_{m\geq 1}$. Moreover, modifying the techniques in [**15**, Section 6] and [**2**, Sections 5 and 6], we can prove the following:

$$\lim_{k\to\infty} Y(N_k, [g_k]) = Y(N_\infty, [g_\infty]).$$

However, the above strict inequality is sufficient for our proof that $Y(N, [g]) < Y_2$.

Unlike each N_k, the noncompact manifold N_∞ satisfies the conditions (i), (ii), and (iii) of Theorem 1.1, and hence, at least heuristically, this suggests that

$$\limsup_{k\to\infty} Y(N_k, [g_k]) = Y(N_\infty, [g_\infty]) \leq Y_2.$$

It should be also pointed out that the scalar-flat, asymptotically flat manifold conformally equivalent to $(N_\infty - \{\text{point}\}, g_\infty)$, will have two singularities created by the two ends of N_∞. Therefore, we can not apply the inverse mean curvature flow technique directly to this manifold. Instead, we apply it to a family of asymptotically flat manifolds without singularities (but with boundary), arising from the manifolds $(N_k^*, g_{k,AF})$ that we describe next.

Fix a point $p_\infty \in N_\infty$ and call its projection to each N_k also by p_∞. Let G_k be the normalized Green's function on N_k for the conformal Laplacian L_{g_k} with the pole at p_∞. Consider the following scalar-flat, asymptotically flat metrics

$$g_{k,AF} := G_k^4 g_k \quad \text{on} \quad N_k^* := N_k - \{p_\infty\}.$$

Using inverse mean curvature flow techniques, we will show

Theorem 4.1.

$$\limsup_{k\to\infty} Y(N_k^*, [g_k]) \leq Y_2.$$

The proof of Theorem 4.1 is given in the next two sections. Some extra difficulties arise for the general case because the infinite Riemannian covering will have infinitely many ends. Fortunately, the method developed here and in Sections 5, 6 is robust enough to handle those cases. The proof will be given in Section 7.

5. Proof of main theorem: simple case

Under the same assumptions of Section 4, consider the normalized minimal positive Green's function G_∞ on (N_∞, g_∞) for the conformal Laplacian L_{g_∞} with the pole at p_∞ (cf. [**34**, Section 1], [**2**, Section 6]). This function is defined on $N_\infty^* := N_\infty - \{p_\infty\}$ and it is essential to control the geometry of $(N_k^*, g_{k,AF})$. All the statements in this section regarding these manifolds will follow from the good understanding we have for G_∞.

We start by collecting the properties G_∞ and G_k in Lemma 5.1. After this, we describe the main idea behind the proof of Theorem 4.1 and state four auxiliary lemmas, whose proofs will be given in the next section. We then prove Theorem 4.1.

For reasons of simplicity, we set the following notation for the rest of the paper. Any quantity Q that depends on $g_{k,AF}$ or g_k will be denoted by $Q_{k,AF}$ or Q_k respectively. If it is obvious from the context which metric is being considered, we will just denote it by Q.

Let γ be the deck transformation of N_∞ corresponding to the homotopy class of $\langle c\rangle \in \pi_1(N)$, that is,

$$N_k = N_\infty / \{\langle \gamma^{jk}\rangle \mid j \in \mathbb{Z}\} \quad \text{for} \quad k \geq 1.$$

For simplicity, we often identify a set in N_∞ with its projection to each N_k and a function on N_k with its lift to N_∞. For instance, G_k denotes both the normalized Green's function on N_k and its lift to N_∞.

Standard arguments developed in [**34**], [**11**], and [**2**], imply the following lemma regarding the functions G_k and G_∞. Here, note that each g_k and g_∞ are metrics of positive constant scalar curvature $Y(N, [g]) > 0$.

Lemma 5.1.

(i) *For all* $k \geq 1$,

$$G_k = \sum_{j \in \mathbb{Z}} G_\infty \circ \gamma^{jk} \quad \text{on } N_\infty.$$

(ii) *For all* $k \geq 2$,

$$0 < G_\infty < G_k < G_1,$$

and G_k *converges uniformly in* C^ℓ *(for every* $\ell \geq 1$*) to* G_∞ *on every compact subset of* N_∞^*.

(iii) *For every compact set* K *of* N_∞^*,

$$\lim_{j \to \infty} \sup \left\{ G_\infty(\gamma^j(x)) \,|\, x \in K \right\} = 0.$$

(iv) *For any open set* O *containing* p_∞*, there exists a constant* L *independent of* k *such that, for all* $k \geq 1$,

$$|\nabla G_k|_k \leq L G_k \quad \text{on } N_k - O.$$

(v)

$$\int_{N_\infty} G_\infty \, dV_\infty < \infty.$$

Take a smooth connected embedded 2-sphere S in N_∞ with $p_\infty \in S$ such that the projection from S into N is injective and its image in N intersects transversely c at one point. Then, consider the embedded 2-spheres

$$F_k := \gamma^{[k/2]}(S) \sqcup \gamma^{[k/2]-k}(S)$$

in N_∞ for each $k \geq 2$. Denote the projection of F_k ($\subset N_\infty$) to each N_k also by F_k. Note that F_k is connected and that

$$\lim_{k \to \infty} \operatorname{dist}_{g_k}(p_\infty, F_k) = \infty.$$

The topological condition on M implies that all non-separating 2-spheres of N_k^* are cobordant to each other. Such 2-spheres correspond to a " neck" on N_k^*. The properties (ii) and (iii) say that $(N_k^*, g_{k,AF})$ has, for k large enough, a very small "neck". This is the content of the following Lemma.

Lemma 5.2. *For every* k*, there exists an area-minimizing, non-separating, connected embedded 2-sphere* S_k *of* $(N_k^*, g_{k,AF})$ *such that*

$$\lim_{k \to \infty} \left| S_k \right|_{k,AF} = 0,$$

where $\left| S_k \right|_{k,AF}$ *denotes the area of* S_k *with respect to* $g_{k,AF}$.

Set X_k to be the metric completion of

$$N_k^* - S_k = N_k - (\{p_\infty\} \cup S_k),$$

and denote the smooth metric extension of $g_{k,AF}$ to X_k by the same symbol. Then, $(X_k, g_{k,AF})$ becomes an asymptotically flat manifold

with minimal boundary consisting of two 2-spheres. This corresponds to "cutting out the small neck" of N_k^*. The advantage of working with X_k rather than N_k^* is that, as can be easily seen, any embedded 2-sphere of X_k bounds a connected region. Therefore, by Theorem 3.4 and Lemma 3.5, the monotonicity of the Hawking quasi-local mass is preserved for each solution of inverse mean curvature flow on $(X_k, g_{k,AF})$, provided that its initial condition is the connected boundary of a connected region. Taking into account that the area $|\partial X_k|_{k,AF}$ of ∂X_k goes to zero as k goes to infinity, we expect to construct a nice family of test functions u_k with

$$\limsup_{k\to\infty} Q_{g_{k,AF}}(u_k) \leq Y_2.$$

Using the inverse mean curvature flow techniques in the same spirit as in [**7**], we have the following result.

Lemma 5.3. *Let (X, h) be an asymptotically flat 3-manifold with minimal boundary ∂X. Assume further that there exists a weak solution for the inverse mean curvature flow, with initial condition a minimal surface Σ_0, that keeps the Hawking quasi-local mass nondecreasing. Then, there exists a function v in $W^{1,2}(X, h)$ such that $0 \leq v \leq 1$ on X and*

$$\frac{\int_X 8|\nabla v|^2 dV_h}{\left(\int_X v^6 dV_h\right)^{1/3}} \leq Y_2 + \theta\left(\left|\partial X\right|\left|\Sigma_0\right|^{-1}\right),$$

where θ is a continuous function independent of (X, h) and vanishing at zero.

An *outermost minimal surface* of an asymptotically flat 3-manifold (X, h) is a minimal surface (possibly disconnected) which encloses all other compact minimal surfaces. Since each N_k^* is not diffeomorphic to $\mathbb{R}^3$, a result due to Meeks, Simon and Yau [**23**] implies that $(X_k, g_{k,AF})$ has a unique outermost minimal surface Ω which is the disjoint union of a finite number of weakly embedded 2-spheres (i.e., limits of uniformly smooth embedded 2-spheres). Moreover, the region exterior to the outermost minimal surface is $\mathbb{R}^3$ minus a finite number of closed balls (see [**13**, Section 4] for details). The next Lemma guarantees that one of the connected components of the outermost minimal surface has area bounded below independently of k.

Lemma 5.4. *The outermost minimal surface of each $(X_k, g_{k,AF})$ has a connected component $\Sigma(k)$ with area bounded below independently of k.*

Remark 5.5. The surface $\Sigma(k)$ is a minimizing hull because it is contained in the outermost minimal surface. If $\Sigma(k)$ is a 2-sphere, then it is the boundary of a precompact connected open set, and hence it can be used as an initial condition for the inverse mean curvature flow (see

Theorem 3.2). If $\Sigma(k)$ is a one-sided $\mathbb{RP}^2$, then it is the limit of uniformly smooth embedded 2-spheres that are minimizing hulls bounding a precompact open connected set. We then take the limit of the weak solutions to inverse mean curvature flow starting at each of these 2-spheres, in order to get a weak solution to the inverse mean curvature flow with initial condition $\Sigma(k)$. This can be done rigorously using the compactness theorem and the a priori bounds proved in [**13**, Sections 2 and 3].

It can be easily seen that Theorem 4.1 is false when M is either S^3 or $S^2 \times S^1$, in which case N is $S^2 \times S^1$ or $(S^2 \times S^1)\#(S^2 \times S^1)$, respectively. The technical reason why this proof doesn't work for those cases is because Lemma 5.4 doesn't hold. It is worthwhile to point out that, among the auxiliary lemmas we state in this section, Lemma 5.4 is the only lemma where the topological restrictions on M play a role.

The function v given by Lemma 5.3 may not vanish on the boundary. Therefore, if we want to apply this result to our sequence of asymptotically flat manifolds $(X_k,\, g_{k,AF})$, we need to multiply the resulting functions by some cut-off function vanishing on the boundary ∂X_k. Otherwise, we could not extend it to be a valid test function on N_k^*. This is a delicate issue because, in general, the multiplication by a cut-off function makes the Yamabe quotient increase. Fortunately, using Lemma 5.1 (iv) and (v), we can prove the following result.

Lemma 5.6. *For every $\varepsilon > 0$, there exists a sequence of Lipschitz functions η_k $(0 \le \eta_k \le 1)$ such that*

$$\eta_k \equiv 0 \quad \text{on } \partial X_k, \qquad \eta_k \equiv 1 \quad \text{outside a compact set of } X_k,$$

and for all k sufficiently large

$$\int_{X_k} |\nabla \eta_k|^2 \, dV_{k,AF} \le \varepsilon, \quad \mathrm{Vol}_{k,AF}\Big(\{\, x \in X_k \mid \nabla \eta_k \neq 0 \,\}\Big) \le \varepsilon.$$

We can now prove Theorem 4.1.

Proof of Theorem 4.1. The proof follows if we find a sequence of test functions u_k vanishing on X_k such that

$$\limsup_{k\to\infty} \frac{\displaystyle\int_{X_k} 8|\nabla u_k|^2 dV_{k,AF}}{\left(\displaystyle\int_{X_k} u_k^6 dV_{k,AF}\right)^{1/3}} \le Y_2.$$

Consider the minimal surface $\Sigma(k)$ given by Lemma 5.4. There exists a solution to the inverse mean curvature flow with initial condition $\Sigma(k)$ (Remark 5.5) and the topological condition on X_k implies the monotonicity of the Hawking quasi-local mass along the solution. From

Lemma 5.3, we have the existence, for all k, of v_k $(0 \le v_k \le 1)$ in $W^{1,2}(X_k, g_{k,AF})$ such that

$$\frac{\displaystyle\int_{X_k} 8|\nabla v_k|^2 dV_{k,AF}}{\left(\displaystyle\int_{X_k} v_k^6 dV_{k,AF}\right)^{1/3}} \le Y_2 + \theta\left(\left|\partial X_k\right|_{k,AF}\left|\Sigma(k)\right|_{k,AF}^{-1}\right).$$

Take any small $\varepsilon > 0$ and let η_k $(0 \le \eta_k \le 1)$ denote the Lipschitz function given in Lemma 5.6 for all sufficiently large k. Consider the admissible nonnegative function in $W^{1,2}(N_k^*,\, g_{k,AF})$ given by

$$u_k := \eta_k v_k.$$

We then have

$$\begin{aligned}
&\int_{X_k} 8|\nabla u_k|^2\, dV_{k,AF} \\
&\le 8\left\{\int_{X_k} |\nabla v_k|^2\, dV_{k,AF}\right. \\
&\quad \left. + 2\sqrt{\int_{X_k} |\nabla v_k|^2\, dV_{k,AF}}\sqrt{\int_{X_k} |\nabla \eta_k|^2\, dV_{k,AF}} + \int_{X_k} |\nabla \eta_k|^2\, dV_{k,AF}\right\} \\
&\le q_k\left(\int_{X_k} v_k^6\, dV_{k,AF}\right)^{1/3} + 4\sqrt{2\varepsilon q_k}\left(\int_{X_k} v_k^6\, dV_{k,AF}\right)^{1/6} + 8\varepsilon,
\end{aligned}$$

where

$$q_k := Y_2 + \theta\Big(\left|\partial X_k\right|_{k,AF}\left|\Sigma(k)\right|_{k,AF}^{-1}\Big) > 0.$$

Following the proof of Lemma 5.3 we see that

$$\int_{X_k} v_k^6\, dV_{k,AF} \ge \left|\Sigma(k)\right|_{k,AF}^{3/2}\left(C_0 + \theta\big(\left|\partial X_k\right|_{k,AF}\left|\Sigma(k)\right|_{k,AF}^{-1}\big)\right) \ge C > 0$$

for some positive constants C_0, C independent of k, where the second inequality comes from combining Lemma 5.2 with the lower bound on $\left|\Sigma(k)\right|_{k,AF}$ (Lemma 5.4), and with the continuity of θ. Hence, using the properties of η_k given by Lemma 5.6, we obtain

$$\begin{aligned}
\frac{\displaystyle\int_{X_k} v_k^6\, dV_{k,AF}}{\displaystyle\int_{X_k} u_k^6\, dV_{k,AF}} &\le \frac{\displaystyle\int_{X_k} v_k^6\, dV_{k,AF}}{\displaystyle\int_{X_k} v_k^6\, dV_{k,AF} - \mathrm{Vol}_{k,AF}\Big(\{\, x \in X_k \mid \nabla\eta_k \ne 0 \,\}\Big)} \\
&\le \frac{C}{C-\varepsilon}
\end{aligned}$$

for some positive constant C independent of k, and therefore

$$\frac{\displaystyle\int_{X_k} 8|\nabla u_k|^2\, dV_{k,AF}}{\left(\displaystyle\int_{X_k} u_k^6\, dV_{k,AF}\right)^{1/3}} \leq q_k \left(\frac{C}{C-\varepsilon}\right)^{1/3} + 4\sqrt{2q_k}\frac{\sqrt{\varepsilon}\, C^{1/6}}{(C-\varepsilon)^{1/3}} + \frac{8\varepsilon}{(C-\varepsilon)^{1/3}}.$$

Combining Lemma 5.4 with the fact that

$$\left|\partial X_k\right|_{AF}$$

converges to zero (Lemma 5.2), we have

$$\lim_{k\to\infty} q_k = Y_2.$$

The proof follows by letting ε go to zero. q.e.d.

6. Proofs of auxiliary lemmas

We first prove Lemma 5.2, then Lemma 5.3, Lemma 5.4, and finally Lemma 5.6.

Proof of Lemma 5.2. Consider the same smooth connected embedded 2-spheres S in N_∞ and F_k in N_k^* that were constructed right before stating Lemma 5.2. Then

$$\left|\gamma^j(S)\right|_{\infty,AF} = \int_S G_\infty^4 \circ \gamma^j\, dA_\infty.$$

Hence, from Lemma 5.1 (ii) and (iii),

$$\lim_{k\to\infty} \left|F_k\right|_{k,AF} = 0.$$

Next, we minimize area in the isotopy class of F_k. The theory developed in [**23**] says that the area minimizer exists and is a smooth disjoint union of 2-spheres and one-sided real projective planes. Moreover, according to [**23**, Section 3], the connected components of the area minimizer can be joined by arbitrarily thin tubes so that the resulting surface is isotopic to F_k. Hence, the area minimizer has to contain a smooth minimal 2-sphere S_k that is non-separating.

The desired assertion follows from the fact that $\left|S_k\right|_{k,AF} \leq \left|F_k\right|_{k,AF}$. q.e.d.

Proof of Lemma 5.3. We use the same idea of [**7**] and construct a test function v whose level sets coincide with the weak solution of the inverse mean curvature flow. This time we have to take into account that X has a minimal boundary ∂X. Let ϕ be the weak solution to the inverse mean curvature flow with initial condition Σ_0 and set $\Sigma_t := \partial\{\phi < t\}$. Like in [**7**], v is defined to be

$$v(x) := \begin{cases} f(0) & \text{if } \phi(x) \leq 0, \\ f(\phi(x)) & \text{if } \phi(x) > 0, \end{cases}$$

where

$$f(t) = \frac{1}{\sqrt{2e^t - e^{t/2}}}.$$

After a simple computation using the coarea formula [**7**, Section 6], we get

$$\int_X |\nabla v|^2 dV_h = \int_0^\infty f'(t)^2 \int_{\Sigma_t} H dA_h \, dt, \tag{4}$$

$$\int_X v^6 \, dV_h \geq \int_0^\infty f(t)^6 \left|\Sigma_t\right|^2 \left(\int_{\Sigma_t} H \, dA_h\right)^{-1} dt. \tag{5}$$

Next, we find an upper bound for $\int_{\Sigma_t} H \, dA_h$.

Lemma 6.1. *For all $t \geq 0$,*

$$\int_{\Sigma_t} H dA_h \leq \sqrt{16\pi\left|\Sigma_0\right|(e^t - e^{t/2})} + \sqrt{16\pi\left|\partial X\right|e^t}.$$

Proof. Using the monotonicity of the Hawking quasi-local mass, we have

$$\begin{aligned}\sqrt{\frac{\left|\Sigma_t\right|}{(16\pi)^3}}\left(16\pi - \int_{\Sigma_t} H^2 \, dA_h\right) &= m_H(\Sigma_t) \\ &\geq m_H(\Sigma_0) \\ &= \sqrt{\frac{\left|\Sigma_0\right|}{16\pi}}.\end{aligned}$$

Hence, it follows from Hölder's inequality that

$$\begin{aligned}\int_{\Sigma_t} H dA_h &\leq \sqrt{\left|\Sigma_t\right| \int_{\Sigma_t} H^2 \, dA_h}, \\ &\leq \sqrt{16\pi\left(\left|\Sigma_t\right| - \sqrt{\left|\Sigma_t\right|\left|\Sigma_0\right|}\right)},\end{aligned}$$

and thus, by Theorem 3.2 (iii),

$$\begin{aligned}\int_{\Sigma_t} H dA_h &\leq \sqrt{16\pi\left|\Sigma_0\right|(e^t - e^{t/2}) + 16\pi\left|\partial X\right|e^t} \\ &\leq \sqrt{16\pi\left|\Sigma_0\right|(e^t - e^{t/2})} + \sqrt{16\pi\left|\partial X\right|e^t}.\end{aligned}$$

q.e.d.

Combining Lemma 6.1 first with (4) and then with (5), we get

$$\int_X |\nabla v|^2 dV_h \leq \sqrt{16\pi\left|\Sigma_0\right|}\left(\int_0^\infty f'(t)^2\sqrt{e^t - e^{t/2}} \, dt + C\left|\partial X\right|\left|\Sigma_0\right|^{-1}\right),$$

for some positive constant C. Then, Theorem 3.2 (iii) implies the following

$$\begin{aligned}\int_X v^6\,dV_h &\geq \int_0^\infty \frac{f(t)^6\big|\Sigma_t\big|^2}{\sqrt{16\pi\big|\Sigma_0\big|(e^t-e^{t/2})}+\sqrt{16\pi\big|\partial X\big|e^t}}\,dt\\ &\geq \big|\Sigma_0\big|^{3/2}\int_0^\infty \frac{f(t)^6e^{2t}}{\sqrt{16\pi(e^t-e^{t/2})}+\sqrt{16\pi\big|\partial X\big|\big|\Sigma_0\big|^{-1}e^t}}dt\\ &= \big|\Sigma_0\big|^{3/2}\left(\int_0^\infty \frac{f(t)^6e^{2t}}{\sqrt{16\pi(e^t-e^{t/2})}}dt+\theta\left(\big|\partial X\big|\big|\Sigma_0\big|^{-1}\right)\right),\end{aligned}$$

where θ is some function independent of the metric. It can be easily seen that θ is continuous and vanishes when $\big|\partial X\big|\big|\Sigma_0\big|^{-1}$ is zero. According to [**7**], we know that

$$\frac{(16\pi)^{2/3}\displaystyle\int_0^\infty f'(t)^2\sqrt{e^t-e^{t/2}}\,dt}{\left(\displaystyle\int_0^\infty f(t)^6e^{2t}\,(e^t-e^{t/2})^{-1/2}dt\right)^{1/3}}=\frac{Y_2}{8},$$

and hence

$$\begin{aligned}&\frac{\displaystyle\int_X|\nabla v|^2dV_h}{\left(\displaystyle\int_X v^6dV_h\right)^{1/3}}\\ &\leq \frac{(16\pi)^{2/3}\displaystyle\int_0^\infty f'(t)^2\sqrt{e^t-e^{t/2}}\,dt}{\left(\displaystyle\int_0^\infty f(t)^6e^{2t}\,(e^t-e^{t/2})^{-1/2}dt\right)^{1/3}}+\theta\left(\big|\partial X\big|\big|\Sigma_0\big|^{-1}\right)\\ &=\frac{Y_2}{8}+\theta\left(\big|\partial X\big|\big|\Sigma_0\big|^{-1}\right),\end{aligned}$$

where θ is a continuous function vanishing at zero and independent of the metric h. q.e.d.

Proof of Lemma 5.4. We argue by contradiction. Suppose that there exists an infinite number of asymptotically flat 3-manifolds $(X_k,\ g_{k,AF})$ (which we still index by k), such that the area of every connected component of the outermost minimal surface Ω_k $(\subset X_k)$ goes to zero when k goes to infinity.

We show the existence of a connected component $\Omega_{k,0}$ of Ω_k for which there exist two subsets $A_k,\ B_k$ of X_k, both intersecting $\Omega_{k,0}$, and such that the distance between A_k and B_k (with respect to $g_{k,AF}$) is bounded from below independently of k. Thus, $\Omega_{k,0}$ will have, for k large enough,

very small area and diameter bounded from below. Moreover, it will be apparent from the construction of A_k and B_k that the complement of both sets in X_k has bounded sectional curvature independently of k. A simple application of the monotonicity formula shows that $\Omega_{k,0}$ cannot have very small area (cf. [**27**], [**28**]).

First, we construct A_k. Denote by $\widehat{N}$ ($\subset N_\infty$) the closure of a fundamental domain of N containing p_∞ and take a positive constant α such that

$$A_k := G_\infty^{-1}[\alpha, +\infty)$$

contains $\widehat{N} - \{p_\infty\}$. Note that this set cannot be diffeomorphic to $\mathbb{R}^3$ minus a finite number of balls. For all k sufficiently large, this set projects injectively on X_k and so we can identify A_k with its projection on X_k. Since the region exterior to Ω_k is topologically trivial, we get that one of its connected components $\Omega_{k,0}$ must intersect A_k.

Define

$$B_k := G_k^{-1}[0, \alpha/2].$$

We now show that, for all k sufficiently large, B_k intersects $\Omega_{k,0}$. Consider

$$s_k := \inf\left\{ \left|\Sigma\right|_k \,\middle|\, \Sigma \text{ is a homotopically non-trivial surface of } N_k \right\},$$

and note that $0 < s_1 \leq s_k$ for all k. For any k large enough satisfying

$$\left|\Omega_{k,0}\right|_{k,AF} = \int_{\Omega_{k,0}} G_k^4 dA_k \leq (\alpha/2)^4 s_1,$$

we must have that

$$\Omega_{k,0} \cap G_k^{-1}[0, \alpha/2] \neq \emptyset.$$

We just need to check that the distance between A_k and B_k is bounded from below independently of k. Let

$$d := \mathrm{dist}_{g_\infty}\left(G_\infty^{-1}[\alpha,\, +\infty),\, G_\infty^{-1}[0, \alpha/2]\right) > 0.$$

Regarding B_k as its lift to X_∞, we have from Lemma 5.1 (ii),

$$B_k \subset G_\infty^{-1}[0, \alpha/2].$$

Therefore, for all k sufficiently large,

$$\mathrm{dist}_{k,AF}(A_k, B_k) \geq \mathrm{dist}_{\infty,AF}(A_k, G_\infty^{-1}[0, \alpha/2]) \geq (\alpha/2)^2 d.$$

q.e.d.

Proof of Lemma 5.6. For any $\delta > 0$, set

$$\eta_{k,\delta} := \min\left\{1, \frac{(G_k - \delta)_+}{\delta}\right\},$$

where $(G_k - \delta)_+$ denotes the positive part of $(G_k - \delta)$. First note that ∂X_k is minimal and its area goes to zero when k goes to infinity. Using

a similar argument to the proof of Lemma 5.4, we obtain that, for all k sufficiently large,

$$\partial X_k \subset G_k^{-1}[0,\, \delta].$$

It suffices to check that, for any $\varepsilon > 0$, we can find a small $\delta > 0$ so that

$$\int_{X_k} |\nabla \eta_{k,\delta}|^2 \, dV_{k,AF} \leq \varepsilon \quad \text{and} \quad \mathrm{Vol}_{k,AF}\Big(\Big\{x \in X_k \,|\, \nabla \eta_{k,\delta} \neq 0\Big\}\Big) \leq \varepsilon$$

for all k sufficiently large.

For any open set O containing p_∞ in N_∞, Lemma 5.1 (iii) and (v) imply that

$$\int_{N_\infty - O} G_\infty^6 \, dV_\infty < \infty.$$

Combining this with Lemma 5.1 (iii), we can choose small $\delta > 0$ so that

$$\int_{G_\infty^{-1}[\delta,\, 2\delta]} G_\infty^6 \, dV_\infty \leq \frac{\varepsilon}{2}.$$

The uniform approximation property of Lemma 5.1 (ii) guarantees that

$$\mathrm{Vol}_{k,AF}\Big(\{x \in X_k \,|\, \nabla \eta_{k,\delta} \neq 0\}\Big) \leq \int_{G_k^{-1}[\delta,\, 2\delta]} G_k^6 \, dV_k < \varepsilon$$

for all k large enough. Likewise, Lemma 5.1 (iv) implies that

$$\begin{aligned}\int_{X_k} |\nabla \eta_{k,\delta}|^2_{k,AF} \, dV_{k,AF} &\leq \int_{G_k^{-1}[\delta,\, 2\delta]} G_k^2 \delta^{-2} |\nabla G_k|^2 \, dV_k \\ &\leq 4L^2 \int_{G_k^{-1}[\delta,\, 2\delta]} G_k^2 \, dV_k.\end{aligned}$$

Arguing as before, we can find a small $\delta > 0$ so that

$$4L^2 \int_{G_k^{-1}[\delta,\, 2\delta]} G_k^2 \, dV_k < \varepsilon$$

for all k sufficiently large. This completes the proof. q.e.d.

7. Proof of main theorem: general case

In this section, we complete the proof of Theorem 2.1. Let L be a closed 3-manifold for which each prime factor contains either a non-separating S^2 or a two-sided $\mathbb{RP}^2$. Then, according to [**12**, Theorem 12.7 and its Proof], L can be expressed as

$$L := \#\ell_1(S^2 \times S^1)\#\ell_2(S^2 \tilde{\times} S^1)\#Q_1\# \cdots \#Q_{\ell_3},$$

where $\ell_1 + \ell_2 + \ell_3 \geq 1$ and each Q_i is a prime closed 3-manifold with a two-sided $\mathbb{RP}^2$ which is homotopy equivalent to $\mathbb{RP}^2 \times S^1$. Furthermore, set

$$N := M\#L,$$

where M is a closed 3-manifold satisfying the conditions (ii) and (iii) in Theorem 2.1. For the proof of the assertion (1) in Theorem 2.1, we assume that M is not diffeomorphic to S^3 and $\ell_1+\ell_2 \geq 1$. For the proof of the assertion (2) in Theorem 2.1, we assume that $\ell_3 \geq 1$. In order to prove Theorem 2.1 it suffices to show that

$$Y(N,[g]) < Y_2$$

for any unit-volume Yamabe metric g on N with $Y(N,[g]) > 0$.

We now describe a suitable infinite Riemannian covering of (N,g). If L has no two-sided $\mathbb{RP}^2$ (i.e., $\ell_3 = 0$), define L_∞ to be the (infinite) universal covering of L. In this case, $\ell_1 + \ell_2 \geq 1$ and M satisfies the conditions (i), (ii) and (iii) in Theorem 2.1. If L has a two-sided $\mathbb{RP}^2$ (i.e., $\ell_3 \geq 1$), then there exists a normal infinite covering $L_\infty \to L$ having a two-sided $\mathbb{RP}^2$ in its fundamental domain and such that every embedded 2-sphere and every two-sided $\mathbb{RP}^2$ in L_∞ separate L_∞. Finally, let (N_∞, g_∞) be the normal infinite Riemannian covering of (N,g) associated to the normal infinite covering L_∞ of L which is topologically

$$N_\infty = \#_1^\infty M \# L_\infty.$$

Before stating the relevant properties of N_∞, we need to introduce some notation.

Fix $p_\infty \in N_\infty$ and consider a sequence of finite Riemannian coverings $\{(N_k, g_k)\}_{k\geq 1}$ of (N,g) satisfying $(N_1, g_1) = (N,g)$ and the following:

(i) N_∞ is an infinite covering of each N_k.
(ii) N_{k+1} is a finite covering of N_k for $k \geq 1$.
(iii) For $k \geq 2$, there exists a fundamental domain of N_k containing p_∞ such that its closure $\widehat{N_k}$ $(\subset N_\infty)$ satisfies

$$\{x \in N_\infty \mid \mathrm{dist}_{g_\infty}(x, p_\infty) \leq k\} \subsetneq \widehat{N_k}.$$

We can assume, without loss of generality, that $\partial\widehat{N_k}$ is smooth and that $\partial\widehat{N_{k+1}}, \partial\widehat{N_k}$ have the same projection on N_k for all $k \geq 2$. Denote by $\widehat{N}$ $(\subset N_\infty)$ the closure of a fundamental domain of N containing p_∞ such that $\partial\widehat{N_2}$ and $\partial\widehat{N}$ have the same projection on N.

The important properties of N_∞ are that every embedded 2-sphere and every two-sided $\mathbb{RP}^2$ in N_∞ separate N_∞, and that $\widehat{N} - \{p_\infty\}$ is not diffeomorphic to $\mathbb{R}^3$ minus a finite number of balls. This last assertion follows from the fact that either M satisfies the conditions (i)-(iii) in Theorem 2.1 or the covering $N_\infty \to N$ has a fundamental domain with a two-sided $\mathbb{RP}^2$.

Let

$$\mathcal{G} := \pi_1(N)/\pi_1(N_\infty) \quad \text{and} \quad \mathcal{G}_k := \pi_1(N_k)/\pi_1(N_\infty) \subset \mathcal{G}$$

denote the groups of deck transformations for the normal covering $N_\infty \to N$ and for each of the normal coverings $N_\infty \to N_k$, respectively.

Here, we identify $\pi_1(N_\infty)$ with its projections to $\pi_1(N)$ and $\pi_1(N_k)$. Set

$$W_n := \{\gamma \in \mathcal{G} \mid \mathrm{dist}_{g_\infty}(\gamma(p_\infty), p_\infty) \geq n\}.$$

Each $\mathcal{G}_k$ has the property that

$$\mathcal{G}_k - \{e\} \subset W_{q(k)},$$

where $\{q(k)\}$ is a sequence of positive integers going to infinity with $q(k) \leq q(k+1)$ for all $k \geq 1$.

We denote the projection of p_∞ to each N_k also by p_∞. Like in Section 4, let G_k be the normalized Green's function on N_k for the conformal Laplacian L_{g_k} with the pole at p_∞. Consider the following scalar-flat, asymptotically flat metrics

$$g_{k,AF} := G_k^4 g_k \quad \text{on} \quad N_k^* := N_k - \{p_\infty\}.$$

Using Aubin's Lemma 3.6,

$$Y(N, [g]) < Y(N_k, [g_k]) < Y(N_{k+1}, [g_{k+1}])$$

for all $k \geq 2$. Therefore, it suffices to prove the following theorem.

Theorem 7.1.

$$\lim_{k\to\infty} Y(N_k^*, [g_{k,AF}]) \leq Y_2.$$

Proof. The strategy for the proof is the same as in Sections 4 and 5.

Consider the normalized minimal positive Green's function G_∞ on N_∞ for L_{g_∞} with the pole at p_∞. This function is defined on $N_\infty^* := N_\infty - \{p_\infty\}$. Note that Lemma 5.1 still holds, with some modifications, in this new setting. For example, Lemma 5.1 (i) becomes

$$G_k = \sum_{\gamma \in \mathcal{G}_k} G_\infty \circ \gamma \quad \text{on } N_\infty \tag{6}$$

for all $k \geq 1$, and Lemma 5.1 (iii) now says that

$$\lim_{n\to\infty} \sup\{G_\infty(\gamma(x)) \mid x \in K,\, \gamma \in W_n\} = 0 \tag{7}$$

for every compact set K of N_∞^*.

The main difference between the case we consider in this section and the one considered in Section 5 is that the number of "necks" in N_k^* increases with k. Nevertheless, we show that these "necks" are very small when k is large enough.

Lemma 7.2. *For any k, there exists a family Γ_k of disjoint embedded surfaces in N_k^* such that each of its connected components is either a 2-sphere or a two-sided $\mathbb{RP}^2$ which minimizes area (with respect to $g_{k,AF}$), and that*

$$\lim_{k\to\infty} \left|\Gamma_k\right|_{k,AF} = 0.$$

Moreover, every embedded 2-sphere and every embedded two-sided $\mathbb{RP}^2$ of $N_k^* - \Gamma_k$ bound a connected region.

Proof. Take a subset S of the connected components of $\partial\widehat{N}$ which projects injectively into N^*, such that every embedded 2-sphere and every embedded two-sided $\mathbb{RP}^2$ of $N - S$ separate. Note that $N - S$ is connected and that S is the disjoint union of connected components $\{S_j\}_{j=1}^{j_0}$, which are either embedded 2-spheres or two-sided real projective planes. From the construction of N_k, (replacing $\{q(k)\}$ by another sequence with the same properties if necessary) there exist subsets

$$I_{j,k} \subset W_{q(k)} \subset \mathcal{G} \quad \text{for} \quad j = 1, \cdots, j_0,$$

such that

$$(I_{1,k}(S_1) \sqcup \cdots \sqcup I_{j_0,k}(S_{j_0})) \subset \partial\widehat{N_k},$$

and every embedded 2-sphere and every two-sided $\mathbb{RP}^2$ of

$$N_k - (I_{1,k}(S_1) \sqcup \cdots \sqcup I_{j_0,k}(S_{j_0}))$$

separate, where

$$I_{j,k}(S_j) := \{\gamma(S_j) \mid \gamma \in I_{j,k}\} \quad \text{for} \quad j = 1, \cdots, j_0.$$

Note also that $N_k - (I_{1,k}(S_1) \sqcup \cdots \sqcup I_{j_0,k}(S_{j_0}))$ is connected. We now check that

$$\lim_{k\to\infty} \left|I_{1,k}(S_1) \sqcup \cdots \sqcup I_{j_0,k}(S_{j_0})\right|_{k,AF} = 0.$$

From (6), we have for $j = 1, \cdots, j_0$,

$$\sum_{\gamma\in\mathcal{G}} \int_{S_j} G_\infty \circ \gamma \, dA_\infty = \int_{S_j} G_1 \, dA_1 < \infty.$$

Hence,

$$\begin{aligned}
\lim_{k\to\infty} \int_{I_{j,k}(S_j)} G_k \, dA_k &= \lim_{k\to\infty} \sum_{\gamma\in\mathcal{G}_k} \int_{I_{j,k}(S_j)} G_\infty \circ \gamma \, dA_\infty \\
&\leq \lim_{k\to\infty} \sum_{\gamma\in W_{q(k)}} \int_{S_j} G_\infty \circ \gamma \, dA_\infty \\
&= 0,
\end{aligned}$$

provided that $q(k)$ goes to infinity. Using (7), we have that

$$\lim_{k\to\infty} \sup\{G_\infty(x) \mid x \in I_{i,k}(S_i)\} = 0 \quad \text{for} \quad j = 1, \cdots, j_0,$$

and thus this implies

$$\begin{aligned}
\lim_{k\to\infty} \left|I_{j,k}(S_j)\right|_{k,AF} &= \lim_{k\to\infty} \int_{I_{j,k}(S_j)} G_k^4 \, dA_k \\
&\leq \lim_{k\to\infty} \int_{I_{j,k}(S_j)} G_k \, dA_k \\
&= 0.
\end{aligned}$$

The proof follows by minimizing area (with respect to $g_{k,AF}$) in the isotopy class of $I_{1,k}(S_1) \sqcup \cdots \sqcup I_{j_0,k}(S_{j_0})$. The existence and regularity

theory in [**23**] says that the area minimizer is, in this case, a disjoint union of 2-spheres and real projective planes. Again, according to [**23**, Section 3], the connected components of the area minimizer can be jointed by arbitrarily thin tubes so that the resulting surface is isotopic to $I_{1,k}(S_1) \sqcup \cdots \sqcup I_{j_0,k}(S_{j_0})$. Hence, the area minimizer has to contain a minimal surface Γ_k ($\subset N_k^*$) consisting of 2-spheres or two-sided real projective planes such that every embedded 2-sphere and every two-sided $\mathbb{RP}^2$ in $N_k^* - \Gamma_k$ separate. q.e.d.

Set X_k to be the metric completion of

$$N_k^* - \Gamma_k = N_k - (\{p_\infty\} \cup \Gamma_k),$$

and denote the smooth metric extension of $g_{k,AF}$ to X_k by the same symbol. Then, $(X_k, g_{k,AF})$ becomes an asymptotically flat manifold with minimal boundary consisting of a disjoint union of 2-spheres and projective planes. Note that Theorem 3.4 and Lemma 3.5 imply the monotonicity of the Hawking quasi-local mass for each solution of the inverse mean curvature flow on $(X_k, g_{k,AF})$, provided that its initial surface is the boundary of a connected region.

Since $\widehat{N} - \{p_\infty\}$ is not diffeomorphic to $\mathbb{R}^3$ minus a finite number of balls, there exists a positive constant α such that $G_\infty^{-1}[\alpha, +\infty)$ has also this property. The proof of Lemma 5.4 can then be easily modified to give the following.

Lemma 7.3. *The outermost minimal surface of each $(X_k, g_{k,AF})$ has a connected component $\Sigma(k)$ with area bounded from below independently of k.*

The argument in the proof of Lemma 5.6 is still applicable to this new setting. The same arguments as in the proof of Theorem 4.1 in Section 4 complete the proof of Theorem 7.1. q.e.d.

References

[1] K. Akutagawa, Spin^c *geometry, the Seiberg-Witten equations and Yamabe invariants of Kähler surfaces*, Interdiscip. Inform. Sci. **5** (1999) 55–72, MR 1693402, Zbl 0982.53039.

[2] K. Akutagawa & B. Botvinnik, *Yamabe metrics on cylindrical manifolds*, Geom. Funct. Anal. **13** (2003) 259–333, MR 1982146.

[3] T. Aubin, *The scalar curvature*, 'Differential Geometry and Relativity', Math. Phys. and Appl. Math., **3**, 5–18, Reidel, Dordrecht, 1976, MR 0433500, Zbl 0345.53029.

[4] ———, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. **55** (1976) 269–296, MR 0431287, Zbl 0336.53033.

[5] ———, *Some Nonlinear Problems in Riemannian Geometry*, Springer Monographs in Mathematics, Springer, 1998, MR 1636569, Zbl 0896.53003.

[6] A. Besse, *Einstein Manifolds*, Springer, 1987, MR 0867684, Zbl 0613.53001.

[7] H. Bray & A. Neves, *Classification of prime 3-manifolds with Yamabe invariant greater than* $\mathbb{RP}^3$, Ann. of Math. **159** (2004) 407–424, MR 2052359, Zbl 1066.53077.

[8] M. Gromov & H.B. Lawson Jr., *The classification of simply connected manifolds of positive scalar curvature*, Ann. of Math. **111** (1980) 423–434, MR 0577131, Zbl 0463.53025.

[9] ———, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Inst. Hautes Études Sci. Publ. Math. **58** (1983) 83–196, MR 0720933, Zbl 0538.53047.

[10] M. Gursky & C. LeBrun, *Yamabe invariants and* Spin^c *structures*, Geom. Funct. Anal. **8** (1998) 965–977, MR 1664788, Zbl 0931.53019.

[11] L. Habermann, *Riemannian Metrics of Constant Mass and Moduli Spaces of Conformal Structures*, Lect. Notes in Math., **1743**, Springer, 2000, MR 1790086, Zbl 0964.58008.

[12] J. Hempel, 3-*Manifolds*, Ann. of Math. Studies, **86**, Princeton Univ. Press, 1976, MR 2098385, Zbl 1058.57001.

[13] G. Huisken & T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom. **59** (2001) 353–437, MR 1916951, Zbl 1055.53052.

[14] M. Ishida & C. LeBrun, *Curvature, connected sums, and Seiberg-Witten theory*, Comm. Anal. Geom. **11** (2003) 809–836, MR 2032500.

[15] O. Kobayashi, *Scalar curvature of a metric with unit volume*, Math. Ann. **279** (1987) 253–265, MR 0919505, Zbl 0611.53037.

[16] N. Koiso, *On the second derivative of the total scalar curvature*, Osaka J. Math. **16** (1979) 413–421, MR 0539596, Zbl 0415.53033.

[17] H.B. Lawson & M.L. Michelsohn, *Spin Geometry*, Princeton Univ. Press, 1989, MR 1031992, Zbl 0688.57001.

[18] C. LeBrun, *Four manifolds without Einstein metrics*, Math. Res. Lett. **3** (1996) 133–147, MR 1386835, Zbl 0856.53035.

[19] ———, *Yamabe constants and the perturbed Seiberg-Witten equations*, Comm. Anal. Geom. **5** (1997) 535–553, MR 1487727, Zbl 0901.53028.

[20] ———, *Kodaira dimension and the Yamabe problem*, Comm. Anal. Geom. **7** (1999) 133–156, MR 1674105, Zbl 0996.32009.

[21] ———, *Einstein metrics and the Yamabe problem*, in 'Trends in mathematical physics' (Knoxville, TN, 1998), 353–376, AMS/IP Stud. Adv. Math., **13**, Amer. Math. Soc., Providence, RI, 1999, MR 1708770, Zbl 1050.53032.

[22] J. Lee & T. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc. **17** (1987) 37–81, MR 0888880, Zbl 0633.53062.

[23] W.H. Meeks III, L. Simon, & S.-T. Yau, *Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature*, Ann. of Math. **116** (1982) 621–659, MR 0678484, Zbl 0521.53007.

[24] M. Obata, *The conjectures on conformal transformations of Riemannian manifolds*, J. Differential Geom. **6** (1971) 247–258, MR 0303464, Zbl 0236.53042.

[25] J. Petean, *Computations of the Yamabe invariant*, Math. Res. Lett. **5** (1998) 703–709, MR 1671183, Zbl 0974.53506.

[26] J. Petean & G. Yun, *Surgery and the Yamabe invariant*, Geom. Funct. Anal. **9** (1999) 1189–1199, MR 1736933, Zbl 0976.53045.

[27] R. Schoen, *Estimates for stable minimal surfaces in three-dimensional manifolds*, in 'Seminar on Minimal Submanifolds', Ann. of Math. Stud., **103**, 111–126, Princeton Univ. Press, 1983, MR 0795231, Zbl 0532.53042.

[28] ———, *Analytic aspects of the harmonic map problem*, in 'Seminar on Nonlinear Partial Differential Equations', Math. Sci. Res. Inst. Publ., **2**, 321–358, Springer, 1984, MR 0765241, Zbl 0551.58011.

[29] ———, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Differential Geom.**20** (1984) 479–495, MR 0788292, Zbl 0576.53028.

[30] ———, *Recent progress in geometric partial differential equations*, in 'Proceedings of the International Congress of Mathematicians' (Berkeley, Calif., 1986), 121–130, Amer. Math. Soc., Providence, RI, 1987, MR 0934219, Zbl 0692.35046.

[31] ———, *Variational theory for the total scalar curvature functional for Riemannian metrics and related topics*, in 'Topics in Calculus of Variations', Lect. Notes in Math., **1365**, 121–154, Springer, 1989, MR 0994021, Zbl 0702.49038.

[32] R. Schoen & S.-T. Yau, *Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature*, Ann. of Math. **110** (1979) 127–142, MR 0541332, Zbl 0431.53051.

[33] ———, *On the structure of manifolds with positive scalar curvature*, Man. Math. **28** (1979) 159–183, MR 0535700, Zbl 0423.53032.

[34] ———, *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math. **92** (1988) 47–71, MR 0931204, Zbl 0658.53038.

[35] ———, *Lectures on Differential Geometry*, in 'Conference Proceedings and Lecture Notes in Geometry and Topology I', International Press, 1994, MR 1333601, Zbl 0830.53001.

[36] S. Stolz, *Positive scalar curvature metrics – Existence and classification questions*, in 'Proceedings of the International Congress of Mathematicians' (Zürich 1994), Birkhäuser, 1995, 625–636, MR 1403963, Zbl 0848.57021.

[37] H. Yamabe, *On a deformation of Riemannian structures on compact manifolds*, Osaka J. Math. **12** (1960) 21–37, MR 0125546, Zbl 0096.37201.

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